MTH 1420, SPRING 2012

SECTION 5.2: DEFINITE INTEGRALS

1. Introduction

In the previous section we looked at approximating the area under a curve. One can extend this notion to finding the *exact* area under a curve, but to do so you need to bring in limits. As you may recall from Calc 1, applying limits can be rather tedious, and one of the main things you should learn from this section is that finding the exact area under a curve by using limits is not much fun at all. This will make us desperate for an easier way, which we shall learn about in the sections to come. Most of the material in this section was presented in section 5.1, but we now introduce some more notation and terminology that will be important in the sections to come.

2. The Definite Integral

Recall that we use sigma notation to indicate when we are adding up a bunch of terms. For example, if x_1, x_2, x_3 are my sample points (suppose they are all right endpoints), then I could approximate the area under a curve with three subintervals as $f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$. Using sigma notation, we write this as

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x = \sum_{i=1}^{3} f(x_i)\Delta x.$$

Definition 1. Let f be a continuous function. Then the <u>definite integral</u> of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where the x_1^*, x_2^*, etc are the sample points in each subinterval (which we usually take to be the right or left endpoint or the midpoint). Note that $\Delta x = \dfrac{b-a}{n}$ and x_i^* lies in the subinterval $[x_{i-1}, x_i]$; so x_1^* is in the interval $[x_0, x_1]$ and x_8^* is in the interval $[x_7, x_8]$.

We call a the <u>lower</u> limit of integration and b the <u>upper</u> limit of integration.

Definition 2. A Riemann Sum is equal to $\sum_{i=1}^{n} f(x_i^*)\Delta x$ without the lint in front. Riemann sums are used to get an approximation for $\int_a^b f(x)dx$, and are exactly what we calculated in the previous section.

Remark. Both a definite integral and a Riemann sum can be negative, if the function f happens to lie below the x-axis.

Example 3. Approximate $\int_0^{2\pi} (x - 2\sin(2x))dx$ by calculating the Riemann sum for $x - 2\sin(2x)$ on the interval $[0, 2\pi]$, using six subintervals and the midpoint as your sample points.

3. Evaluating a definite integral using the definition

As annoying as it may be to use Riemann sums to approximate the definite integral, actually calculating a definite integral can be much more annoying and terrible. We did one example in the previous section that was tedious, and we will now do one that is even worse.

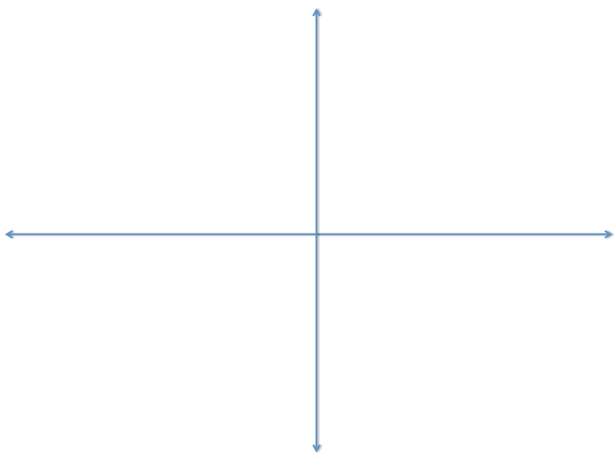
Example 4. Evaluate $\int_1^5 (x^2 - 3x)dx$.

Exercise 5. Evaluate $\int_0^3 (2x + 3)dx$.

4. Evaluating an integral using geometry

Sometimes you can calculate a definite integral by using the fact that it gives you the exact area between the curve and the x-axis, as well as area formulas from geometry.

Example 6. Find $\int_0^9 f(x)dx$, where $f(x) = \begin{cases} \frac{2}{3}x - 2 & \text{on } [0,3] \\ \sqrt{9-(x-6)^2} & \text{on } [3,9] \end{cases}$.



5. Properties of the Definite Integral

Using our knowledge of area, as well as the definition of the definite integral, we can prove the following properties:

(a) $\int_a^b f(x)dx = -\int_b^a f(x)dx$.
(b) $\int_a^a f(x)dx = 0$.
(c) $\int_a^b (f(x) \pm g(x))dx = -\int_b^a f(x)dx \pm \int_b^a g(x)dx$.
(d) $\int_a^b c \cdot f(x)dx = c\int_b^a f(x)dx$ where c is any constant.
(e) $\int_a^b c dx = c(b-a)$ where c is any constant.
(f) $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$.

Justification of (a), (c), and (e):

Exercise 7. Use similar ideas to the justifications above to explain (b), (d) and (f). A short equation or diagram might be enough for some of them.